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Strict Inequalities for Some Critical Exponents in Two-Dimensional Percolation

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For 2D percolation we slightly improve a result of Chayes and Chayes to the effect that the critical exponent β for the percolation probability is *strictly* less than 1. The same argument is applied to prove that if $\mathscr{S}(\varphi) := \{(x, y): x = r\cos\theta,\ y = r\sin\theta$ for some $r \geqslant 0$, or $\theta \leqslant \varphi\}$ and $\beta(\varphi) := \lim_{p \downarrow p_c}[\log(p - p_c)]^{-1} \log P_{\rm cr}\ \{O$ is connected to ∞ by an occupied path in $\mathscr{S}(\varphi)\}$, then $\beta(\varphi)$ is strictly decreasing in φ on $[0, 2\pi]$. Similarly, $\lim_{n \to \infty}[-\log n]^{-1} \log P_{\rm cr}\ \{O$ is connected by an occupied path in $\mathscr{S}(\varphi)$ to the exterior of $[-n, n] \times [-n, n]$ is strictly decreasing in φ on $[0, 2\pi]$.

KEY WORDS: Percolation; critical exponent for percolation probability; percolation in sectors; strict inequalities.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a matching pair of periodic graphs $(\mathscr{G}, \mathscr{G}^*)$ imbedded in $\mathbb{R}^2$ (see Ref. 7, Chapter 2 for this terminology). For instance $\mathscr{G}$ could be $\mathbb{Z}^2$ and $\mathscr{G}^*$ the graph with the same vertex set as $\mathbb{Z}^2$, but with edge set the edges of $\mathbb{Z}^2$ plus all the diagonals of the unit squares $[n_1, n_1 + 1] \times [n_2, n_2 + 1]$, n_1, $n_2 \in \mathbb{Z}$. Now choose each vertex of $\mathscr{G}$, independently of all others, occupied with probability p and vacant with probability $q := 1 - p$. This model is called site-percolation on $\mathscr{G}$. In the above-mentioned example we obtain site-percolation on $\mathbb{Z}^2$, and this will be our prime example. It is well known that bond percolation on $\mathbb{Z}^2$ or many other two-dimensional lattices can also be formulated as site-percolation on one graph of a matching pair (see Ref. 7, Section 3.1). Our results here apply to all such examples that satisfy condition (1.4) below. P_p will denote the probability measure on the configurations of occupied and vacant sites of $\mathscr{G}$, corresponding to the above

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model. A path on $\mathscr{G}$ is called occupied if all its vertices are occupied, and $A \to B$ (in C) means that there exists an occupied path from some vertex in A to some vertex in B (with all its vertices in C). W, the occupied cluster of O, is given by

$$W = \{y: O \to y\}$$

(Here we tacitly assume that O, the origin, is a vertex of $\mathscr{G}$; if this is not the case, one can take for W the occupied cluster of any fixed site w_0). $\# W$ denotes the number of vertices in W, and the percolation probability is given by

$$\theta(p) = P_p\{\# W = \infty\}$$

The critical probability is

$$p_c = \sup\{p: \theta(p) = 0\}$$

and we write P_{cr} for P_{p_c}.

Chayes and Chayes[1] recently proved that for some constant[2] $C_1 = C_1(\mathscr{G}) > 0$,

$$\theta(p) - \theta(p_c) \geqslant C_1(p - p_c), \qquad p > p_c \tag{1.1}$$

(The result of Ref. 1 holds in all dimensions.) If one makes the commonly accepted assumptions

$$\theta(p_c) = 0 \tag{1.2}$$

and

$$\beta := \lim_{p \downarrow p_c} \frac{1}{\log(p - p_c)} \log \theta(p) \quad \text{exists} \tag{1.3}$$

then (1.1) implies $\beta \leqslant 1$. Our main result implies that $\beta < 1$ under the following condition:

> The x or y axis is an axis of symmetry for $\mathscr{G}$ and $\mathscr{G}^*$, and in addition, $\mathscr{G}$ and $\mathscr{G}^*$ are invariant under rotation (around O) over a fixed angle $\theta_0 \in (0, \pi)$. (1.4)

Our result is formulated in such a way that we do not have to assume (1.3). It is known, though, that (1.2) follows from (1.4) [see after (2.9) below].

[2] C_i will denote a strictly positive and finite constant whose precise value is of no significance for our purposes. The value of C_i may change from one appearance to another.

Theorem 1. If the two-dimensional matching pair $(\mathcal{G}, \mathcal{G}^*)$ satisfies (1.4), then there exist constants $0 < C_1 = C_1(\mathcal{G}) < \infty$ and $\beta = \beta(\mathcal{G}) < 1$ such that

$$\theta(p) \geqslant C_1(p - p_c)^{\beta}, \qquad p > p_c \tag{1.5}$$

It is possible to use the proof of Theorem 1 for several related strict inequalities. We make the following definitions:

$$\mathcal{S}(\varphi) = \{(x, y): x = r\cos\theta,\ y = r\sin\theta \text{ for some } r \geqslant 0, 0 \leqslant \theta \leqslant \varphi\}$$

($\mathcal{S}$ is a sector of the plane),

$$\theta(p, \varphi) = P_p\{O \to \infty \text{ in } \mathcal{S}(\varphi)\}$$

[note $\theta(p) = \theta(p, 2\pi)$],

$$S(n) = [-n, n] \times [-n, n]$$

$$\mathring{S}(n) = (-n, n) \times (-n, n) = \text{interior of } S(n)$$

$$S^c(n) = \mathbb{R}^2 \setminus S(n) = \text{complement of } S(n)$$

$$\partial S(n) = \text{(topological) boundary of } S(n)$$

$$\zeta(\varphi, n) = P_{\text{cr}}\{O \to S^c(n) \text{ in } \mathcal{S}(\varphi)\}$$

$$\pi(n) = \zeta(2\pi, n) = P_{\text{cr}}\{O \to S^c(n)\}$$

Theorem 2. For $0 \leqslant \varphi_1 < \varphi_2 \leqslant 2\pi$ there exist constants $0 < C_i = C_i(\varphi_1, \varphi_2, \mathcal{G}) < \infty$ such that

$$\frac{\theta(p, \varphi_1)}{\theta(p, \varphi_2)} \leqslant C_2(p - p_c)^{C_3}, \qquad p > p_c \tag{1.6}$$

$$\frac{\zeta(n, \varphi_1)}{\zeta(n, \varphi_2)} \leqslant C_4 n^{-C_5} \tag{1.7}$$

Moreover, for some $0 < C_i = C_i(\mathcal{G}) < \infty$

$$\frac{\theta(p, \varphi)}{\theta(p)} \leqslant C_6(p - p_c)^{C_7} \text{ uniformly in } p > p_c, \qquad 0 \leqslant \varphi < 2\pi \tag{1.8}$$

and

$$\frac{\zeta(n, \varphi)}{\pi(n)} \leqslant C_8 n^{-C_9} \text{ uniformly in } 0 \leqslant \varphi < 2\pi \tag{1.9}$$

Remark 1. If one assumes that

$$\beta(\varphi) := \lim_{p \downarrow p_c} \frac{1}{\log(p - p_c)} \log \theta(p, \varphi) \quad \text{exists}$$

then (1.6) and (1.8) say that $\beta(\varphi)$ is strictly decreasing in φ on $[0, 2\pi)$, with a jump at 2π. Relations (1.7) and (1.9) have a similar interpretation for

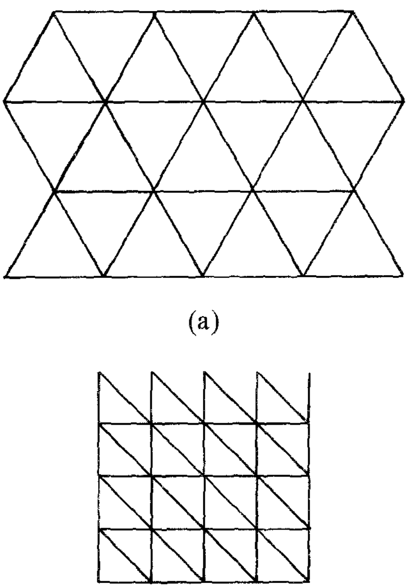
$$\lim_{n \to \infty} \frac{1}{\log n} \log \zeta(n, \varphi) \tag{1.10}$$

provided this limit exists.

Remark 2. If the limits in (1.10) exist, they depend on the imbedding of $\mathscr{G}$ into $\mathbb{R}^2$. For example, (1.7) implies that

$$P_{\mathrm{cr}}\{O \to S^c(n) \text{ in the first quadrant}\}$$

has different asymptotic behavior for the two imbeddings of the triangular lattice shown in Figs. 1a and 1b.



(a)

(b)

Fig. 1.

In fact, $\zeta(n, \pi/2)$ for the imbedding of Fig. 1b corresponds to $\zeta(n, \pi/3)$ for the imbedding of Fig. 1a. It is not clear how one should define a limit analogous to (1.10) that is independent of the imbedding. It is even less clear that universality of the φ dependence will hold for these limits, once one has a good definition. J. T. Chayes and L. Chayes (private communication) have suggested that the definitions given above will lead to universality, provided the graphs can be imbedded in such a way that the x and y axis are axes of symmetry and

$$P_{\text{cr}}\{\text{the left and right edges of the square } S(n) \text{ are connected by an occupied path in } S(n)\} \to \tfrac{1}{2} \quad (n \to \infty)$$

and the same limit holds for connections between the top and the bottom edge.

Remark 3. For $\mathscr{G} = \mathbb{Z}^2$, $\beta(\pi)$ should equal the exponent β_s of Christou and Stinchcombe[4] for the percolation probability of sites near the edge for percolation in a half-plane. Various approximations for β_s are given in Ref. 4.

Remark 4. Results quite similar to Theorems 1 and 2 hold when $\mathscr{S}(\varphi)$ is replaced by the more general sector

$$\mathscr{S}(\varphi, \varphi_0) := \{(x, y) \colon x = r\cos\theta,\ y = r\sin\theta \text{ for some } r \geqslant 0,\ \varphi_0 \leqslant \theta \leqslant \varphi_0 + \varphi\}$$

in the definitions.

2. PROOF OF THEOREM 1 FOR SITE PERCOLATION ON $\mathbb{Z}^2$

For simplicity *we always take $\mathscr{G} = \mathbb{Z}^2$ in this section.* This is the special example mentioned in the introduction. In the next section we explain what additional arguments are needed to cover more general situations.

The proof of (1.1) in Ref. 1 was based on the differential inequality

$$\theta(p) \leqslant \frac{1}{p}\,[\theta^2(p) + \theta(p)\,\theta'(p)] \tag{2.1}$$

We shall obtain (1.5) by improving (2.1) to

$$\theta(p) \leqslant \frac{2}{p}\,[\theta^2(p) + \theta(p)\,\theta'(p)(p - p_c)^\varepsilon], \qquad p > p_c \tag{2.2}$$

for some $\varepsilon > 0$. Clearly, (2.2) will imply (1.5) with $\beta = 1 - \varepsilon$. As explained in Remark 5, it is easy to see that (2.2) will follow once we have Proposition 1 below. Here a circuit on $\mathscr{G}^*$ is a path $(v_0, e_1^*, v_1,..., e_n^*, v_n)$ on $\mathscr{G}^*$ with $v_i \neq v_j$ for all $i \neq j$, except for $v_0 = v_n$ (see Ref. 7, pp. 11 and 29 for details).

Proposition 1. Let x be a vertex of $\mathscr{G}$ (and hence of $\mathscr{G}^*$) and $\mathscr{C}^*$ a circuit on $\mathscr{G}^*$ through x, with all its vertices other than x vacant. Then for $p > p_c$ and $p - p_c$ sufficiently small

$$\begin{aligned} &P_p\{\exists \text{ two disjoint occupied paths on } \mathscr{G} \text{ from neighbors} \\ &\quad \text{of } x \text{ to } \infty \text{ and outside } \mathscr{C}^*\} \\ &\quad \leqslant \frac{2}{p}(p-p_c)^\varepsilon\, \theta(p)\, P_p\{\exists \text{ an occupied path on } \mathscr{G} \text{ from} \\ &\quad \text{a neighbor of } x \text{ to } \infty \text{ and outside } \mathscr{C}^*\} \end{aligned} \tag{2.3}$$

Remark 5. To obtain (2.2) from this, one follows the proof of Ref. 1. The x will be the last pivotal point (or the last articulation point) for the event $\{O \to \infty\}$ if and only if there exists a circuit C^* on $\mathscr{G}^*$ through x with O in its interior and with all its vertices other than x vacant, and if, in addition, there exist three disjoint occupied paths on $\mathscr{G}$, one from O to a neighbor of x in the interior of C^* and two from a neighbor of x to ∞ in the exterior of C^*. By taking C^* to be a minimal circuit through x with O in its interior and with all its vertices other than x vacant, we can apply Lemma 1 instead of Eqs. (11) and (12) in Ref. 1 to estimate for fixed $\mathscr{C}^*$

$$\begin{aligned} &P_p\{x \text{ is the last pivotal site for the event } O \to \infty \\ &\quad \text{and the above minimal circuit } C^* \text{ equals } \mathscr{C}^*\} \\ &\quad = P_p\{\text{a neighbor of } x \text{ is connected to } O \text{ inside } \mathscr{C}^* \\ &\qquad \text{and } \mathscr{C}^* \text{ is minimal vacant circuit through } x \text{ and} \\ &\qquad \text{surrounding } O \text{ (except possibly at } x)\} \\ &\qquad \cdot P_p\{\exists \text{ two disjoint occupied paths from neighbors of} \\ &\qquad x \text{ to } \infty, \text{ outside } \mathscr{C}^*\} \\ &\quad \leqslant P_p\{x \text{ is pivotal for the event } O \to \infty \text{ and the minimal} \\ &\qquad \text{circuit } C^* \text{ equals } \mathscr{C}^*\} \frac{2}{p}(p-p_c)^\varepsilon\, \theta(p) \end{aligned}$$

After summing over all possibilities $\mathscr{C}^*$ for C^*, one can complete the proof of (2.2) exactly as in Ref. 1. ∎

Unfortunately, we must become more technical to explain the proof of Proposition 1. First we need the crossing probabilities of rectangles. On $\mathscr{G} = \mathbb{Z}^2$ an occupied horizontal crossing (or occupied crossing in the

1-direction) of the rectangle $[a_1, a_2] \times [b_1, b_2]$ $(a_i, b_i \in \mathbb{Z})$ is an occupied path $(v_0, e_1, \ldots, e_n, v_n)$ on $\mathscr{G}$ with v_0 on the left edge $\{a_1\} \times [b_1, b_2]$ and v_n on the right edge $\{a_2\} \times [b_1, b_2]$, while all other vertices v_i, $1 \leqslant i \leqslant n-1$, lie in the interior $(a_1, a_2) \times (b_1, b_2)$. Occupied vertical crossings (or occupied crossings in the 2-direction) on $\mathscr{G}$ are defined similarly. Finally, vacant crossings on $\mathscr{G}^*$ are defined by replacing "occupied path on $\mathscr{G}$" by "vacant path on $\mathscr{G}^*$." Now define for $i = 1, 2$

$$\begin{aligned}
&\sigma((a, b); i, p)\\
&\quad = \sigma((a, b); i, p, \mathscr{G})\\
&\quad = P_p\{\exists \text{ occupied crossing on } \mathscr{G} \text{ of } [0, a] \times [a, b] \text{ in the } i\text{-direction}\}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\sigma^*((a, b); i, p)\\
&\quad = \sigma^*((a, b); i, p, \mathscr{G})\\
&\quad = P_p\{\exists \text{ vacant crossing on } \mathscr{G}^* \text{ of } [0, a] \times [0, b] \text{ in the } i\text{-direction}\}
\end{aligned} \tag{2.5}$$

The principal fact we need is that crossing probabilities of rectangles that are not too large (with respect to the correlation length) nor too elongated are bounded away from zero. To make this precise, we first define the correlation length

$$L(p) := \min\{n\colon \sigma((n, n); 1, p) \geqslant 1 - \varepsilon_0, \qquad p > p_c$$

Here ε_0 is some small, strictly positive number whose precise value is not important here. The most important property is that $\varepsilon_0 > 0$ can be chosen such that there exist constants δ_k for which

$$\begin{aligned}
\sigma((kn, n); 1, p) \geqslant \delta_k, \qquad \sigma((n, kn); 2, p) \geqslant \delta_k\\
\sigma^*((kn, n); 1, p) \geqslant \delta_k, \qquad \sigma^*((n, kn); 2, p) \geqslant \delta_k
\end{aligned} \tag{2.6}$$

uniformly in $n \leqslant L(p)$ (for any fixed $k \geqslant 1$). It is shown in Ref. 8, Section 2 that such a choice of ε_0 is possible, and Corollary 2 of Ref. 8 also justifies calling $L(p)$ the correlation length (see also Ref. 2, Proposition 3.2 for another justification). Note that the continuity of σ in p implies that for $p = p_c$, (2.6) will hold uniformly for all n. This can also be proved directly.[10-12] Thus, at p_c the correlation length should be taken as infinity. In fact, the correlation length diverges as $p \to p_c$, and another important property we shall need is

$$L(p) \geqslant C_1 \, |p - p_c|^{-1} \tag{2.7}$$

for some $C_1 > 0$. Equations (4.5) and (4.6) of Ref. 8 imply $L(p) \geqslant C_1 |p - p_c|^{-1/2}$, which would suffice for our purposes here, but Chayes *et al.*[3] have given a much more direct and simple argument for (2.7). Finally, we note that if one combines four crossings to construct a circuit in an annulus as in Refs. 10–12, then one obtains for $k \geqslant 2$

$$P_p\{\exists \text{ occupied circuit on } \mathscr{G} \text{ surrounding } S(n) \text{ in } S(kn) \setminus S(n)\} \geqslant (\delta_4)^4, \qquad n \leqslant L(p) \tag{2.8}$$

$$P_p\{\exists \text{ vacant circuit on } \mathscr{G}^* \text{ surrounding } S(n) \text{ in } S(kn) \setminus S(n)\} \geqslant (\delta_4)^4, \qquad n \leqslant L(p) \tag{2.9}$$

We note here that one easily obtains from (2.9) that for $p = p_c$ there are infinitely many vacant circuits surrounding O, a.e., $[P_{cr}]$; this implies $\theta(p_c) = 0$ (compare Refs. 10 and 11 or Ref. 7, p. 178).

Another ingredient of our proof is an inequality for "disjoint occurrence of two events." A first version of this was proved by Hammersley[5] [cf. Ref. 5, Eq. (25)], and the method of proof has been rediscovered repeatedly (see Ref. 13 for references). To formulate the inequality, we have to bring in an independent copy of our percolation system. More specifically, let $\mathscr{V}$ be the vertex set of $\mathscr{G}$ and set

$$\Omega = \Omega' = \prod_{v \in \mathscr{V}} \{-1, +1\}$$

A typical point ω (ω') of Ω (Ω') is a sequence $\{\omega(v)\}_{v \in \mathscr{V}}$ ($\{\omega'(v)\}_{v \in \mathscr{V}}$). The value $\omega(v) = +1$ (-1) corresponds to v being occupied (vacant). P_p is the product measure on Ω with

$$P_p\{\omega(v) = +1\} = p = 1 - P_p\{\omega(v) = -1\}, \qquad v \in \mathscr{V}$$

We define P'_p in the same way as the product measure on Ω' with

$$P'_p\{\omega'(v) = +1\} = p = 1 - P_p\{\omega'(v) = -1\}$$

For any event $B \subset \Omega$ write B' for its copy in Ω', i.e.,

$$B' = \{\omega' \in \Omega' : \exists \omega \in B \text{ such that } \omega'(v) = \omega(v) \text{ for all } v \in \mathscr{V}\} \tag{2.10}$$

For a fixed $\bar{\omega} \in \Omega$ and $K \subset \mathscr{V}$, $[\bar{\omega}]_K$ denotes the cylinder

$$[\bar{\omega}]_K = \{\omega \in \Omega : \omega(v) = \bar{\omega}(v), v \in K\}$$

Similarly for fixed $\bar{\omega}' \in \Omega'$

$$[\bar{\omega}']_K = \{\omega' \in \Omega' : \omega'(v) = \bar{\omega}'(v), v \in K\}$$

We say that two events A and B of Ω occur disjointly if $A \circ B$ occurs, where

$$A \circ B := \{\omega \in \Omega: \exists K, L \subset \mathscr{V} \text{ such that } K \cap L = \varnothing \\ \text{and } [\omega]_K \subset A, [\omega]_L \subset B\}$$

This terminology should be reasonably intuitive; we interpret $[\omega]_K \subset A$ as "A occurs because of the coordinates of ω in K"; $A \circ B$ then is the event that A and B occur because of disjoint sets of coordinates. We define in a similar way for two events A in Ω and B' in Ω' the event $A \circ B'$ in $\Omega \times \Omega'$ by

$$A \circ B' = \{(\omega, \omega') \in \Omega \times \Omega': \exists K, L \subset \mathscr{V} \text{ such that} \\ K \cap L = \varnothing \text{ and } [\omega]_K \subset A, [\omega']_L \subset B'\}$$

We state the desired inequality as a separate lemma.

Lemma 2. If A and B are increasing events of Ω, each depending on finitely many coordinates only, and B' is the copy of B in Ω' as defined in (2.10), then

$$P_p\{A \circ B\} \leqslant P_p \times P'_p\{A \circ B'\} \tag{2.11}$$

(Here $P_p \times P'_p$ is the product measure of P_p and P'_p on $\Omega \times \Omega'$.)

Proof. We can represent the increasing event A as a finite union $\bigcup A_i$, with A_i of the form

$$A_i = \{\omega \in \Omega: \omega(v) = 1 \text{ for } v \in K_i\}$$

The cylinders A_i can be chosen maximal, or equivalently the $K_i \subset \mathscr{V}$ can be chosen minimal, with the property $A_i \subset A$. If B is also written as a union $\bigcup B_j$ of maximal cylinders in B, with

$$B_j = \{\omega \in \Omega: \omega(v) = 1 \text{ for } v \in L_j\}$$

then it is easy to see that[13]

$$A \circ B = \bigcup_{i,j} (A_i \circ B_j) = \bigcup_{\substack{i,j \\ K_i \cap L_j = \varnothing}} (A_i \circ B_j) = \bigcup_{\substack{i,j \\ K_i \cap L_j = \varnothing}} (A_i \cap B_j)$$

Therefore, (2.11) is immediate from the inequality (3.6) in Ref. 13. [Instead of (3.6) in Ref. 13 one can also use the results of Campanino and Russo, McDiarmid, or Rüschendorf cited there.] ∎

We can now outline the proof of (2.3), at least when the restriction that the paths lie outside $\mathscr{C}^*$ is dropped. (This can be viewed as a degenerate case when $\mathscr{C}^*$ consists of the vertex x only.) By periodicity we may assume $x = O$. Let $\mathscr{A}_i$ be the annulus $S(2^i)\backslash S(2^{i-1})$. An Ω-occupied (Ω'-occupied) path is a path $(v_0, e_1, ..., e_n, v_n)$ on $\mathscr{G}$ with $\omega(v_i) = 1$ $[\omega'(v_i) = 1]$ for $0 \leqslant i \leqslant n$. Now let

$$\mathscr{E}_j = \{\exists \text{ an } \Omega\text{-occupied circuit surrounding } S(2^{3j-1}) \text{ in } \mathscr{A}_{3j} \text{ and an } \Omega\text{-occupied circuit surrounding } S(2^{3j+1}) \text{ in } \mathscr{A}_{3j+2}\} \tag{2.12}$$

and let

$$J = \{j \geqslant 1: 2^{3j+2} \leqslant L(p) \text{ and } \mathscr{E}_j \text{ occurs}\}$$

$\mathscr{E}'$ and J' are defined in the same way for Ω'-occupied circuits, and we further define

$$N = \text{cardinality of } J \cap J'$$

Our first step will be the easy estimate

$$P_p \times P'_p\{N \leqslant -C_2 \log(p - p_c)\} \leqslant 1/2 \tag{2.13}$$

for a suitable $C_2 > 0$ and $p - p_c$ sufficiently small. We now take

$$A = B = \{\text{a neighbor of } O \text{ is connected to } \infty \text{ by an occupied path outside } \mathscr{C}^*\} \tag{2.14}$$

The left-hand side of (2.3) is now $P_p\{A \circ B\}$. Even though A and B depend on infinitely many sites, Lemma 2 and an obvious limit procedure show that the left-hand side of (2.3) is at most

$$P_p \times P'_p\{A \circ B'\} \leqslant 2P_p \times P'_p\{A \circ B' \text{ and } N \geqslant -C_2 \log(p - p_c)\} \tag{2.15}$$

The inequality here follows from (2.13) and the Harris-FKG inequality (Ref. 7, Proposition 4.1) because $A \circ B'$ and $\{N \geqslant -C_2 \log(p - p_c)\}$ are both increasing events in $\Omega \times \Omega'$. We shall estimate (2.15) by conditioning on J, J' and on certain circuits in $\mathscr{A}_{3j}$ and $\mathscr{A}_{3j+2}$, $j \in J \cap J'$. When $\mathscr{E}_j$ occurs, let $\mathscr{C}_{3j}$ be the smallest Ω-occupied circuit surrounding $S(2^{3j-1})$ in $\mathscr{A}_{3j}$, and $\mathscr{D}_{3j+2}$ the largest Ω-occupied circuit surrounding $S(2^{3j+1})$ in $\mathscr{A}_{3j+2}$. Define $\mathscr{C}'_j$ and $\mathscr{D}'_j$ in a similar way as extremal Ω'-occupied circuits when $\mathscr{E}'_j$ occurs (see Fig. 2 for an illustration when $\mathscr{E}_j \cap \mathscr{E}'_j$ occurs). The existence of such smallest circuits $\mathscr{C}_i$, $\mathscr{C}'_i$ and largest circuits $\mathscr{D}_i$, $\mathscr{D}'_i$ can be demonstrated by the method of Lemma 1 of Ref. 6 or Proposition 2.3 of Ref. 7. For any circuit $\mathscr{C}$ define

$$\mathring{\mathscr{C}} = \text{interior of } \mathscr{C}, \qquad \mathscr{C}^e = \text{exterior of } \mathscr{C}, \qquad \bar{\mathscr{C}} = \mathring{\mathscr{C}} \cup \mathscr{C}$$

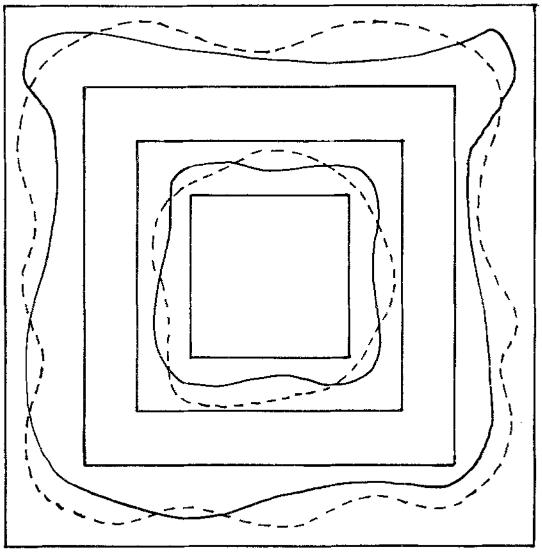


Fig. 2. Illustration of $\mathscr{E}_j \cap \mathscr{E}'_j$. The squares (starting at the innermost square, but not drawn to scale) are $S(2^{3j-1})$, $S(2^{3j})$, $S(2^{3j+1})$, and $S(2^{3j+2})$. The solid circuits are $\mathscr{C}_{3j}$ and $\mathscr{D}_{3j+2}$ and the dashed circuits are $\mathscr{C}'_{3j}$ and $\mathscr{D}'_{3j+2}$.

Then on $\mathscr{E}_j$ we necessarily have $\mathscr{C}_{3j} \subset \mathring{\mathscr{D}}_{3j+2}$, and it follows from the method of Lemma 1 in Ref. 6 or Proposition 2.3 in Ref. 7 that conditionally on $\mathscr{E}_j$, $\mathscr{C}_{3j}$, and $\mathscr{D}_{3j+2}$ the families

$$\{\omega(v): v \in \mathring{\mathscr{D}}_{3j+2} \cap \mathscr{C}^{e}_{3j}\} \tag{2.16}$$

and

$$\{\omega(v): v \in \mathring{\mathscr{C}}_{3j} \cup \mathscr{D}^{e}_{3j+2}\} \cup \{\omega'(u): u \in \mathscr{V}\}$$

are independent. Moreover, the conditional distribution of the family in (2.16) is equal to the unconditional distribution P_p. Now condition on the set of indices J and the occupied circuits $\mathscr{C}_{3j}$, $\mathscr{D}_{3j+2}$, $j \in J$. For the time being drop the restriction concerning $\mathscr{C}^*$ in (2.3) and in (2.14). Let J consist of $j(1) < j(2) < \cdots < j(v)$. Then A occurs if and only if there exist the following collection of Ω-occupied paths (see Fig. 3):

an Ω-occupied path r_0 from a neighbor of O to $\mathscr{C}_{3j(1)}$ that lies in $\mathring{\mathscr{C}}_{3j(1)}$ except for its endpoint on $\mathscr{C}_{3j(1)}$ (2.17)

an Ω-occupied path $s_{j(i)}$ from $\mathscr{C}_{3j(i)}$ to $\mathscr{D}_{3j(i)+2}$ that lies in $\mathring{\mathscr{D}}_{3j(i)+2} \cap \mathscr{C}^{e}_{3j(i)}$ except for its endpoints on $\mathscr{C}_{3j(i)}$ and $\mathscr{D}_{3j(i)+2}$, $i = 1,..., v$ (2.18)

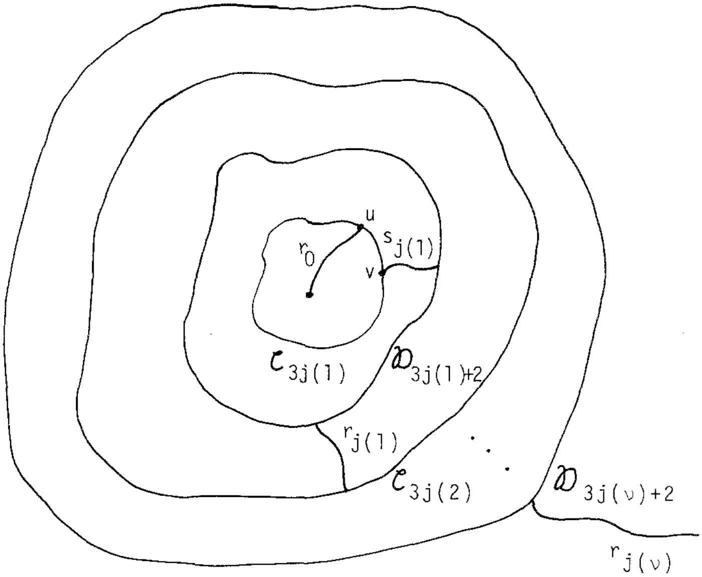


Fig. 3. Given the occupied $\mathscr{C}$'s and $\mathscr{D}$'s, a neighbor of O is connected to ∞ if and only if there exist occupied paths $r_{j(i)}$ and $s_{j(i)}$ as indicated. Note that the endpoint u of r_0 on $\mathscr{C}_{3j(1)}$ and the initial point v of $s_{j(1)}$ are connected by a piece of $\mathscr{C}_{3j(1)}$.

an Ω-occupied path $r_{j(i)}$ from $\mathscr{D}_{3j(i)+2}$ to $\mathscr{C}_{3j(i+1)}$ that lies in $\mathring{\mathscr{C}}_{3j(i+1)} \cap \mathscr{D}^{\mathrm{e}}_{3j(i)+2}$ except for its endpoints on $\mathscr{C}_{3j(i+1)}$ and $\mathscr{D}_{3j(i)+2}$, $i = 1,..., \nu - 1$ (2.19)

an Ω-occupied path $r_{j(\nu)}$ from $\mathscr{D}_{3j(\nu)+2}$ to ∞ in $\mathscr{D}^{\mathrm{e}}_{3j(\nu)+2}$ except for its initial point on $\mathscr{D}_{3j(\nu)+2}$ (2.20)

It is obvious that such Ω-occupied paths must exist for A to occur. In the opposite direction, once such paths exist, they can be connected by pieces of the Ω-occupied circuits $\mathscr{C}_{3j}$ and $\mathscr{D}_{3j+2}$ to make an Ω-occupied path from a neighbor of O to ∞ (see Fig. 3). In exactly the same way, we see that B' occurs if and only if there exist Ω'-occupied paths $r'(i)$ and $s'(i)$ as in (2.17)–(2.20) with $\mathscr{C}$ and $\mathscr{D}$ replaced by $\mathscr{C}'$ and $\mathscr{D}'$ and $j(i)$ by $j'(i)$. For $A \circ B'$ to occur, we must even be able to pick the $\{r_{j(i)}, s_{j(i)}\}$ disjoint from the $\{r'_{j'(i)}, s'_{j'(i)}\}$. We shall only insist on s_j being disjoint from s'_j when $j \in J \cap J'$. This then leads to the following inequality whenever the cardinality of $J \cap J'$ is at least $-C_2 \log(p - p_c)$:

$$P_p \times P'_p\{A \circ B' \text{ and } N \geqslant -C_2 \log(p-p_c) \mid J, J', \mathscr{C}_{3j(i)}, \mathscr{D}_{3j(i)+2}, j(i) \in J, \mathscr{C}'_{3j'(l)}, \mathscr{D}'_{3j'(l)+2}, j'(l) \in J'\}$$
$$\leqslant P_p \times P'_p\{r_0, r_{j(i)}, s_{j(i)} \text{ exist as in (2.17)–(2.20) and their analogues } r'_0, r'_{j'(l)}, \text{ and } s'_{j'(l)} \text{ exist in such a way that } s_j \text{ is disjoint from } s'_j \text{ when } j \in J \cap J' \mid J, J', \mathscr{C}_{3j(i)}, \mathscr{D}_{3j(i)+2}, j(i) \in J, \mathscr{C}'_{3j'(l)}, \mathscr{D}'_{3j'(l)+2}, j'(l) \in J'\} \tag{2.21}$$

When $J \cap J'$ contains fewer than $-C_2 \log(p-p_c)$ indices, then the left-hand side of (2.21) is zero. Now, by the independence statement at (2.16) and its analogue for primed quantities, the right-hand side of (2.21) can be written as the product of the following five factors:

$$P_p\{\text{all the } r\text{'s required by (2.17), (2.19), (2.20) exist} \mid J, \mathscr{C}_{3j(i)}, \mathscr{D}_{3j(i)+2}, j(i) \in J\} \tag{2.22}$$

$$\prod_{j \in J \setminus J'} P_p\{s_j \text{ exists as required by (2.18)} \mid J, \mathscr{C}_{3j}, \mathscr{D}_{3j+2}\} \tag{2.23}$$

$$P_p\{\text{all the } r' \text{ required by the analogous of (2.17), (2.19), and (2.20) exist} \mid J', \mathscr{C}'_{3j'(l)}, \mathscr{D}'_{3j'(l)+2}, j'(l) \in J'\} \tag{2.24}$$

$$\prod_{j' \in J' \setminus J} P'_p\{s'_{j'} \text{ exists as resuired by the analogue of (2.18)} \mid J', \mathscr{C}'_{3j'}, \mathscr{D}'_{3j'+2}\} \tag{2.25}$$

$$\prod_{j \in J' \cap J} P_p \times P'_p\{s_j \text{ and } s'_j \text{ exist and can be chosen disjoint} \mid J, J', \mathscr{C}_{3j}, \mathscr{D}_{3j+2}, \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \tag{2.26}$$

Assume now that we can prove the following estimate for some constant $0 \leqslant \lambda < 1$:

$$P_p \times P'_p\{s_j \text{ and } s'_j \text{ exist and can be chosen disjoint} \mid J, J', \mathscr{C}_{3j}, \mathscr{D}_{3j+2}, \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\}$$
$$\leqslant \lambda P_p\{s_j \text{ as required in (2.18) exists} \mid J, \mathscr{C}_{3j}, \mathscr{D}_{3j+2}\} \cdot P'_p\{s'_j \text{ as required by the analogue of (2.18) exists} \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \tag{2.27}$$

We then obtain that

$$P_p \times P'_p\{A \circ B \text{ and } N \geqslant -C_2 \log(p-p_c)\}$$
$$\leqslant [\text{product of (2.22)–(2.25)}]\ \lambda^{-C_2 \log(p-p_c)}$$
$$\cdot \prod_{j \in J \cap J'} P_p\{s_j \text{ as required in (2.18) exist} \mid J, \mathscr{C}_{3j}, \mathscr{D}_{3j+2}\}$$
$$\cdot \prod_{j \in J \cap J'} P_p\{s'_j \text{ as required by the analogue of (2.18)}$$
$$\text{exists} \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\}$$
$$= \lambda^{-C_2 \log(p-p_c)} P_p\{A \mid J, \mathscr{C}_{3j(i)}, \mathscr{D}_{3j(i)+2}, j(i) \in J\}$$
$$\cdot P'_p\{B' \mid J', \mathscr{C}'_{3j'(l)}, \mathscr{D}'_{3j'(l)+2}, j'(l) \in J'\}$$

Taking expectations with respect to J, J' and all the $\mathscr{C}$, $\mathscr{D}$, $\mathscr{C}'$, and $\mathscr{D}'$, we finally get

$$P_p\{A \circ B\} \leqslant 2\lambda^{-C_2 \log(p-p_c)} P_p\{A\}\ P'_p\{B'\}$$
$$\leqslant \frac{2}{p} (p-p_c)^{C_2 |\log \lambda|}$$
$$\cdot P_p\{\text{a neighbor of } O \text{ is connected to } \infty\}\ \theta(p) \qquad (2.28)$$

since

$$P'_p\{B'\} = P_p\{B\} \leqslant \frac{1}{p}\, \theta(p)$$

To complete the proof of (2.3) without the restriction concerning $\mathscr{C}^*$ we must therefore prove (2.13) and (2.27), which we do in the next two lemmas.

Lemma 3. The result (2.13) holds.

Proof. This is easy by virtue of (2.8) and the independence of the ω and the ω'

$$P_p \times P'_p\{\mathscr{E}_j \cap \mathscr{E}'_j\} = [P_p\{\mathscr{E}_j\}]^2 \geqslant (\delta_4)^{16}$$

In addition, the number of j with $2^{3j+2} \leqslant L(p)$ is at least $-C_3 \log(p-p_c)$, by virtue of (2.7). Since the $\mathscr{E}_j \cap \mathscr{E}'_j$, $j = 1, 2,\ldots$, are independent, (2.13) now follows with

$$C_2 = \tfrac{1}{2}(\delta_4)^{16}\, C_3$$

from Chebyshev's inequality for the binomial distribution, provided $L(p)$ is sufficiently large (or equivalently p sufficiently close to p_c). ∎

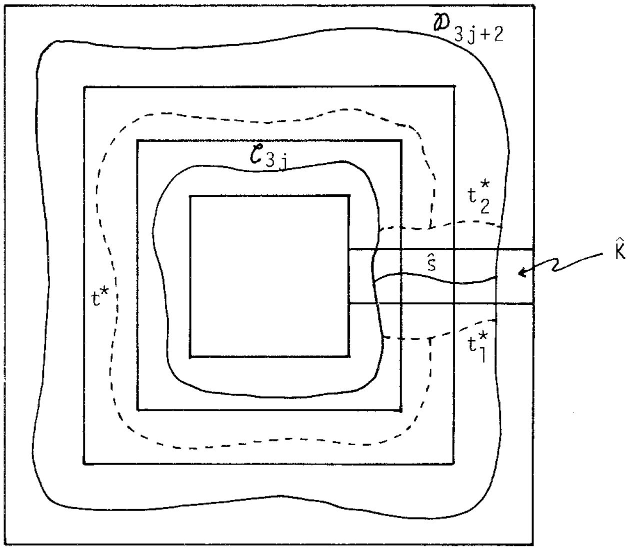


Fig. 4. (—) Ω-Occupied paths and (– –) Ω-vacant paths (except at endpoints on $\mathscr{C}_{3j}$ and $\mathscr{D}_{3j+2}$). Any occupied connection from $\mathscr{C}_{3j}$ to $\mathscr{D}_{3j+2}$ has to lie between t_1^* and t_2^*. Only $\hat{K}$ is shown, but not K_1, K_2.

Lemma 4. The result (2.27) is valid.

Proof. Clearly it suffices to prove

$$P_p \times P'_p\{\text{there exist an } \Omega\text{-occupied connection } s_j \text{ as required in (2.18) and an } \Omega'\text{-occupied connection } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2} \text{ as in the analogue of (2.18), but any two such } s_j \text{ and } s'_j \text{ intersect} \mid J, J', \mathscr{C}_{3j}, \mathscr{D}_{3j+2}, \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \geqslant C_4 > 0 \tag{2.29}$$

(We can then take $\lambda = 1 - C_4$.) To prove (2.29), we show that there is a strictly positive probability that s_j exists, but that it is forced to pass through a certain corridor K, and similarly for s'_j and a corridor K'. Moreover, we choose these corridors such that any $s_j \subset K$ and $s'_j \subset K'$ must intersect. For instance, we can take $K = [2^{3j-1}, 2^{3j+2}] \times [-2^{3j-1}, 2^{3j-1}]$ by assuming the existence of an Ω-occupied path $\hat{s}$ connecting $\mathscr{C}_{3j}$ to $\mathscr{D}_{3j+2}$, two Ω-vacant paths t_1^*, t_2^* on $\mathscr{G}^*$ (in obvious terminology) connecting $\mathscr{C}_{3j}$ to $\mathscr{D}_{3j+2}$, and an Ω-vacant path t^* connecting t_1^* to t_2^* in $\mathscr{A}_{3j+1}$ as indicated in Fig. 4 (the endpoints of t_1^*, t_2^* on $\mathscr{C}_{3j}$ and $\mathscr{D}_{3j+2}$ will not be vacant). The paths t_1^*, $\hat{s}$, and t_2^* in Fig. 4 are pieces of horizontal crossings of

$$K_1 := [2^{3j-1}, 2^{3j+2}] \times [-2^{3j-1}, -2^{3j-1} + 2^{3j-2}]$$
$$\hat{K} := [2^{3j-1}, 2^{3j+2}] \times [-2^{3j-1} + 2^{3j-2}, 2^{3j-1} - 2^{3j-2}]$$
$$K_2 := [2^{3j-1}, 2^{3j+2}] \times [2^{3j-1} - 2^{3j-2}, 2^{3j-1}]$$

respectively. $\hat{s}$ can serve as s_j in this situation, but no occupied s_j can intersect any of the vacant paths t^*, t_1^*, t_2^*, and therefore any such s_j must lie "between t_1^* and t_2^*." It is easy to see that given $\mathscr{C}_{3j}$ and $\mathscr{D}_{3j+2}$, the probability of the occurrence of $\hat{s}$, t^*, t_1^*, t_2^* as above in the Ω-system is at least

$$\begin{aligned} &P_p\{\exists \text{ occupied horizontal crossing of } \hat{K}\} \\ &\quad \cdot P_p\{\exists \text{ vacant horizontal crossing of } K_1\} \\ &\quad \cdot P_p\{\exists \text{ vacant horizontal crossing of } K_2\} \\ &\quad \cdot P_p\{\exists \text{ vacant circuit in } \mathscr{A}_{3j+1} \text{ surrounding } S(2^{3j})\} \\ &\geqslant (\delta_{16})^3 (\delta_4)^4 \end{aligned} \tag{2.30}$$

[by (2.6), (2.9), and the Harris-FKG inequality]. We do not write out the details for the corridor K', but Fig. 5 should convince the reader that the

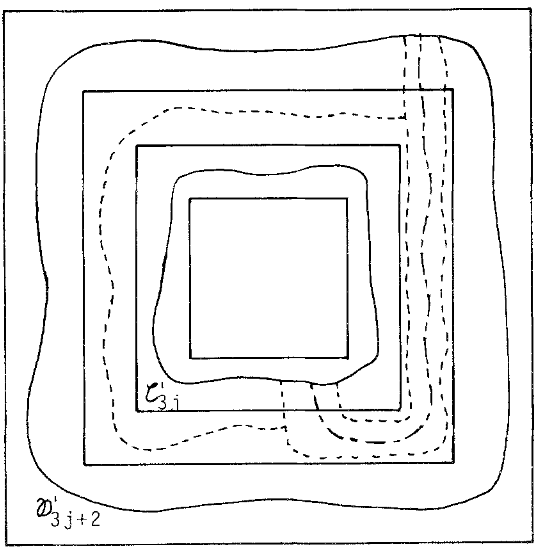


Fig. 5. (—) Ω'-Occupied paths and (- -) Ω'-vacant paths (except at their endpoints on $\mathscr{C}'_{3j}$ and $\mathscr{D}'_{3j+2}$). (- - -) A candidate path for s_j'. Any such path must intersect the $\hat{s}$ of Fig. 4.

existence of suitable Ω'-vacant paths will force s'_j to contain an Ω'-occupied vertical crossing of $[2^{3j}, 2^{3j+1}] \times [-2^{3j-1}, 2^{3j-1}]$. Any such s'_j must intersect any s_j connecting $\mathscr{C}_{3j}$ to $\mathscr{D}_{3j+2}$ in the situation of Fig. 4. We leave it to the reader to verify that the existence of paths such as in Fig. 5 has a conditional P'_p probability (given $\mathscr{C}'_{3j}$ and $\mathscr{D}'_{3j+2}$) that is bounded away from zero. This, together with (2.30) and the independence of the Ω and Ω' systems, implies (2.29). ∎

Proof of Proposition 1. We now show what changes have to be made in the proof of (2.3) in the general case when the restriction of the paths to the exterior of $\mathscr{C}^*$ is imposed. Again we assume $x = O$. We define A and B as in (2.14). We also define

$$i_0 = \max\{i: \mathscr{C}^* \text{ intersects } \partial S(2^i)\}$$

where $\partial S(n)$ denotes the (topological) boundary of $S(n)$. We deal separately with the two cases

$$i_0 \leqslant \frac{\log L(p)}{2 \log 2} \tag{2.31}$$

and [cf. (2.7)]

$$i_0 > \frac{\log L(p)}{2 \log 2} \geqslant C_5 \log(p - p_c)^{-1} - C_6 \tag{2.32}$$

If (2.31) holds, we can use essentially the same proof as before. We define $\mathscr{E}_j$ again by (2.12), but only for $3j > i_0 + 1$, and simularly for $\mathscr{E}'_j$. Also, $\mathscr{C}_{3j}$, $\mathscr{C}'_{3j}$, $\mathscr{D}_{3j+2}$, and $\mathscr{D}'_{3j+2}$ are defined as before, as long as $3j > i_0 + 1$. This time we take

$$J = \{j: 3j > i_0 + 1,\ 2^{3j+2} \leqslant L(p), \text{ and } \mathscr{E}_j \text{ occurs}\}$$

$$J' = \{j': 3j' > i_0 + 1,\ 2^{3j'+2} \leqslant L(p), \text{ and } \mathscr{E}'_{j'} \text{ occurs}\}$$

N is still the cardinality of $J \cap J'$, and the rest of the proof needs no change. We merely need to observe that (2.13) still holds, by the same proof as in Lemma 3, because the number of j with $3j > i_0 + 1$, $2^{3j+2} \leqslant L(p)$ is still at least $-C_3 \log(p - p_c)$ if (2.31) holds.

If (2.32) holds, we use simply

$$P_p\{A \circ B\} \leqslant P_p \times P'_p\{A \circ B'\} \leqslant P_p \times P'_p\{A \cap B'\} = P_p\{A\}\, P'_p\{B'\}$$

(by Lemma 2 and $A \circ B' \subset A \times B$), and then we prove

$$P'_p\{B'\} = P_p\{B\} \leqslant \frac{2}{p} (p - p_c)^{\varepsilon}\, \theta(p), \qquad p > p_c \tag{2.33}$$

Clearly these two inequalities will imply (2.3), so that it suffices to prove (2.33). Note that by (2.14), $P_p\{B\}$ is the probability that a neighbor of O is connected to ∞ outside $\mathscr{C}^*$, while $p^{-1}\theta(p)$ is the probability that a neighbor of O is connected to ∞. Thus, (2.33) says that [under (2.32)] the restriction on the connecting path to lie outside $\mathscr{C}^*$ cuts down the probability by a factor $2(p-p_c)^\varepsilon$.

We can ignore the Ω system for (2.33). We therefore do not need $\mathscr{E}_j$, but we define $\mathscr{E}'_j$ as before if $3j \leqslant i_0 - 2$. Thus we set, analogously to (2.12), for $3j \leqslant i_0 - 2$,

$$\begin{aligned}\mathscr{E}'_j = \{&\exists \text{ an } \Omega'\text{-occupied circuit surrounding } S(2^{3j-1}) \\ &\text{in } \mathscr{A}_{3j} \text{ and an } \Omega'\text{-occupied circuit surrounding} \\ &S(2^{3j+1}) \text{ in } \mathscr{A}_{3j+2}\}\end{aligned}$$

$$J' = \{j\colon j \geqslant 1,\ 3j \leqslant i_0 - 2,\ 2^{3j+2} \leqslant L(p), \text{ and } \mathscr{E}'_j \text{ occurs}\}$$

On $\mathscr{E}'_j$, $\mathscr{C}'_{3j}$ is the smallest Ω'-occupied circuit in $\mathscr{A}_{3j}$ that surrounds $S(2^{3j-1})$, while $\mathscr{D}'_{3j+2}$ is the largest Ω'-occupied circuit in $\mathscr{A}_{3j+2}$ that surrounds $S(2^{3j+1})$. All this is entirely the same as before. Again, if B' occurs, then there must exist paths as in (2.17)–(2.20), now with primes attached to the appropriate entities, and with the added requirement that all these paths lie outside $\mathscr{C}^*$. Consequently, $P'_p\{B' \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}, j \in J'\}$ is bounded by the product of the following two factors:

$$\begin{aligned}&P'_p\{\text{all the } r' \text{ paths required by the analogues of (2.17),} \\ &\quad \text{(2.19), (2.20) exist} \mid J', \mathscr{C}'_{3j(i)}, \mathscr{D}'_{3j(i+2)}, j(i) \in J'\} \qquad (2.34)\end{aligned}$$

$$\begin{aligned}\prod_{j\in J'} &\{\exists\ \Omega'\text{-occupied path } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2} \\ &\text{that lies in } \mathscr{C}^{*\mathrm{e}} \cap \mathring{\mathscr{D}}'_{3j+2} \cap (\mathscr{C}'_{3j})^{\mathrm{e}} \text{ except for} \\ &\text{its endpoints on } \mathscr{C}'_{3j} \text{ and } \mathscr{D}'_{3j+2} \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\}\end{aligned}$$

One can now complete the proof of (2.3) almost as before, if one proves the following analogues of (2.13) and (2.27):

$$P'_p\{\text{cardinality of } J' \leqslant -C_2 \log(p-p_c)\} \leqslant \tfrac{1}{2} \qquad (2.36)$$

and there exists a constant $\lambda < 1$ such that

$$\begin{aligned}&P'_p\{\exists\ \Omega'\text{-occupied path } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2} \text{ that lies} \\ &\quad \text{in } \mathscr{C}^{*\mathrm{e}} \cap \mathring{\mathscr{D}}'_{3j+2} \cap (\mathscr{C}'_{3j})^{\mathrm{e}} \text{ except for its endpoints} \\ &\quad \text{on } \mathscr{C}'_{3j} \text{ and } \mathscr{D}'_{3j+2} \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \\ &\quad \leqslant \lambda P'_p\{\exists\ \Omega'\text{-occupied path } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2} \text{ that} \\ &\qquad \text{lies in } \mathring{\mathscr{D}}'_{3j+2} \cap (\mathscr{C}'_{3j})^{\mathrm{e}} \text{ except for its endpoints on} \\ &\qquad \mathscr{C}'_{3j} \text{ and } \mathscr{D}'_{3j+2} \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \qquad (2.37)\end{aligned}$$

[Note that the left- and right-hand sides of (2.37) differ only in the restriction $s'_j \subset \mathscr{C}^{*e}$.] To obtain (2.3), or rather (2.33), from these we merely have to observe that still

$$\begin{aligned} &P'_p\{r'_0, r'_j, \text{ and } s'_j \text{ exist as required by the analogues of} \\ &\quad (2.17)\text{–}(2.20) \text{ for the } \Omega' \text{ system} \mid J, \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}, j \in J'\} \\ &\leqslant P'_p\{\exists\, \Omega'\text{-occupied path from a neighbor of } O \\ &\quad \text{to } \infty \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}, j \in J'\} \end{aligned}$$

(with all the conditions that the paths lie outside $\mathscr{C}^*$ dropped on both sides).

The proof of (2.13) can be copied almost verbatim for (2.36) [recall that we assume (2.32) now].

Finally we prove (2.37). Its left-hand side is at most

$$\begin{aligned} &P'_p\{\exists\, \Omega'\text{-occupied path } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2} \text{ that does} \\ &\quad \text{not intersect } \mathscr{C}^* \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \end{aligned}$$

Note also that there exists an Ω'-occupied path from $\mathscr{C}'_{3j}$ to $\mathscr{D}'_{3j+2}$ that (with the exception of its endpoints) lies in $\mathring{\mathscr{D}}'_{3j+2} \cap (\mathscr{C}'_{3j})^e$ if and only if there exists any Ω'-occupied path from $\mathscr{C}'_{3j}$ to $\mathscr{D}'_{3j+2}$. As in (2.29) it therefore suffices to prove

$$\begin{aligned} &P'_p\{\exists\, \Omega'\text{-occupied path } s'_j \text{ from } \mathscr{C}'_{3j} \text{ to } \mathscr{D}'_{3j+2}, \text{ but all} \\ &\quad \text{such paths intersect } \mathscr{C}^* \mid J', \mathscr{C}'_{3j}, \mathscr{D}'_{3j+2}\} \geqslant C_4 > 0 \qquad (2.38) \end{aligned}$$

and the proof of (2.38) is analogous to that of (2.29). This time we prove (2.38) by forcing s'_j to lie between Ω'-vacant paths in such a way that s'_j must cross $\mathscr{C}^*$. The choice of these Ω'-vacant paths depends on $\mathscr{C}^*$. Note that $3j+2 \leqslant i_0$, so that there exists a piece ρ^* of $\mathscr{C}^*$ that connects O to $\partial S(2^{3j+1})$. *A fortiori* ρ^* crosses $\mathscr{A}_{3j+1}$ (see Figs. 6 and 7). If $\rho^* \cap \mathscr{A}_{3j+1}$ contains a vertical crossing of $T := [2^{3j}, 2^{3j+1}] \times [-2^{3j}, 2^{3j}]$ as in Fig. 6, then s'_j is forced to intersect ρ^* as soon as there exist Ω'-vacant horizontal crossings t_1^* of

$$T_1 := [2^{3j}, 2^{3j+1}] \times [-2^{3j}, -2^{3j} + 2^{3j-2}]$$

and t_2^* of

$$T_2 := [2^{3j}, 2^{3j+1}] \times [2^{3j} - 2^{3j-2}, 2^{3j}]$$

and an Ω'-vacant connection t^* of t_1^* and t_2^* in $\mathscr{A}_{3j+1} \setminus \hat{T}$, where

$$\hat{T} = [2^{3j}, 2^{3j+1}] \times [-2^{3j} + 2^{3j-2}, 2^{3j} - 2^{3j-2}]$$

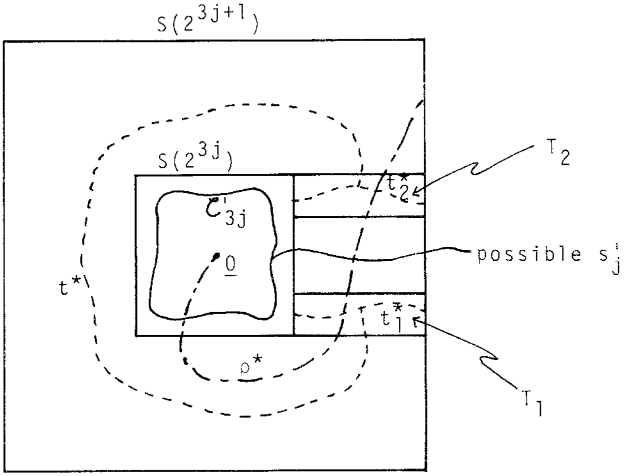


Fig. 6. Illustration of Ω'-vacant paths t_1^*, t_2^*, t^* that force s_j' to intersect ρ^*, where ρ^* contains a vertical crossing of T. (- - -) The path ρ^*.

(Note that $T = T_1 \cup \hat{T} \cup T_2$.) In this situation (2.38) follows by an estimate similar to (2.30). The same argument works if ρ^* contains a horizontal crossing of $T(\pi/2)$ or $T(3\pi/2)$, or a vertical crossing of $T(\pi)$, where $T(\varphi)$ is the image of T after (counterclockwise) rotation over an angle φ.

If ρ^* does not contain a crossing as above of any of T, $T(\pi/2)$, $T(\pi)$, or $T(3\pi/2)$, then one easily checks that ρ^* cannot intersect T as well as

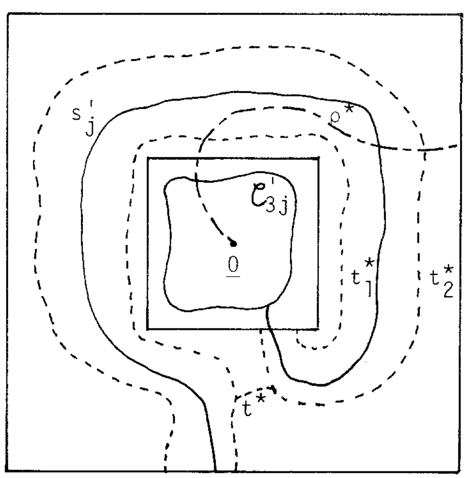


Fig. 7. Illustration of Ω'-vacant paths t_1^*, t_2^*, t^* that force s_j' to intersect ρ^*, where ρ^* does not intersect $T(\pi) \cup T(3\pi/2)$. (- - -) The path ρ^*. (—) A possible s_j'.

$T(\pi)$, nor $T(\pi/2)$ as well as $T(3\pi/2)$. [E.g., in the first case ρ^* would have to cross $T(\pi/2)$ or $T(3\pi/2)$.] In particular, ρ^* can intersect only two of the four rectangles. Assume, for the sake of argument, that ρ^* does not intersect $T(\pi) \cup T(3\pi/2)$, so that ρ^* contains a piece that connects the union of the top and right edges of $S(2^{3j})$ with $\partial S(2^{3j+1})$ in $\mathscr{A}_{3j+1} \backslash T(\pi) \cup T(3\pi/2)$. Figure 7 now illustrates how one can choose Ω'-vacant paths t_1^*, t_2^*, and t^* to force s_j' to wind around from $T(3\pi/2)$ to $T(3\pi/2)$ so that it must intersect ρ^*. We leave it to the reader to show that the P_p'-probability for the existence of paths t_1^*, t_2^*, t^*, and s_j' is bounded away from zero [by virtue of (2.6)], so that (2.38) holds in all situations. ∎

3. COMMENTS ON THE REMAINING PROOFS

Comments to Theorem 1. What goes wrong with the proof of Section 2 when $\mathscr{G}$ is not $\mathbb{Z}^2$? Nothing, as long as $\mathscr{G}$ is planar. However, if $\mathscr{G}$ is nonplanar, the proofs of Lemma 4 and (2.37) break down, because s_j' may cross s_j or ρ^* without having a vertex in common with s_j or ρ^*. Such intersections of paths are allowed in $A \circ B'$. For A and B' to occur disjointly, we merely require that they occur because of disjoint collections of sites of $\mathscr{G}$. The proof of Theorem 1 does, however, go through essentially unchanged for any planar graph $\mathscr{G}$. Only very minor technicalities may need to be changed for a planar $\mathscr{G}$ that is not $\mathbb{Z}^2$, due to the fact that a horizontal crossing of $[a_1, a_2] \times [b_1, b_2]$ with a_i, $b_i \in \mathbb{Z}$ does not necessarily have its endpoints on the left and right edges of this rectangle. Such a crossing will therefore be a path from $[a_1 - \Lambda, a_1] \times [b_1, b_2]$ to $[a_2, a_2 + \Lambda] \times [b_1, b_2]$ for some constant $\Lambda = \Lambda(\mathscr{G})$ (which is an upper bound for the lengths of the edges of $\mathscr{G}$). The arguments in Ref. 7 have been carried out in this generality, and we shall regard the case of planar $\mathscr{G}$ as proven, without going into further technicalities. The restriction to planar $\mathscr{G}$ is, however, serious. It would rule out, for instance, the matching graph of $\mathbb{Z}^2$ and the covering graph of $\mathbb{Z}^2$. The latter corresponds to bond percolation on $\mathbb{Z}^2$ (see Ref. 7, Sections 3.1 and 2.5).

It is easy to save (2.3) by going over to the planar modification $\mathscr{G}_{\text{pl}}$ of $\mathscr{G}$ as explained in Ref. 7, Section 2.3. This is done by adding a so-called central vertex in each face of the mosaic, $\mathscr{M}$ say, on which $(\mathscr{G}, \mathscr{G}^*)$ is based, and which is close-packed for $\mathscr{G}$. These central vertices are taken occupied with probability 1. One obtains a similar planar modification $\mathscr{G}_{\text{pl}}^*$ by inserting a vacant central vertex in all the faces of $\mathscr{M}$ that are close-packed in $\mathscr{G}^*$. Now the proof of Section 2 can be applied to $\mathscr{G}_{\text{pl}}$ to give us (2.3) on $\mathscr{G}_{\text{pl}}$. This does not immediately give us (2.2), though. What we do obtain form (2.3) on $\mathscr{G}_{\text{pl}}$, by the proof of Ref. 1, is

$$\theta(p) \leqslant \frac{2}{p}\Big[\theta^2(p) + (p-p_c)^\varepsilon\,\theta(p) \cdot \sum_{x \in \mathscr{G}_{\rm pl}} P_p\{\text{two neighbors of } x \text{ are connected by occupied paths on } \mathscr{G}_{\rm pl} \text{ to } O \text{ and } \infty\text{, respectively, and there exists a circuit } \mathscr{C}^* \text{ on } \mathscr{G}^*_{\rm pl} \text{ all of whose vertices other than } x \text{ are vacant, and } O \in \text{interior of } \mathscr{C}^*\}\Big] \qquad (3.1)$$

(compare Ref. 7, Proposition 2.1). Any vertex x of $\mathscr{G}_{\rm pl}$ with the properties between the braces on the right-hand side of (3.1) is pivotal for the event $\{O \to \infty\}$. However, the sum over x on the right-hand side of (3.1) is not equal to $\theta'(p)$ [as was the case in Ref. 1, Eq. (7) and following lines]. $\theta'(p)$ is only the sum over those x that are vertices of $\mathscr{G}$ itself, since the other (central) vertices of $\mathscr{G}_{\rm pl}$ are always occupied—the status of these latter vertices is not influenced by p [compare Eq. (4.22) in Ref. 7]. We nevertheless will obtain a version (2.2) by showing that if x is a central vertex of $\mathscr{G}_{\rm pl}$ in the face F of the mosaic $\mathscr{M}$ on which $(\mathscr{G}, \mathscr{G}^*)$ is based, then there exists a vertex y of $\mathscr{G}$ on the perimeter of F such that

$$P_p\{x \text{ has the properties in (3.1)}\} \leqslant C_7 \left(\frac{p}{1-p}\right)^M P_p\{y \text{ has the properties in (3.1)}\} \qquad (3.2)$$

where M is an upper bound for the number of vertices on the perimeter of any face of $\mathscr{M}$. Thus, (3.1) yields

$$\theta(p) \leqslant \frac{2}{p}\left[\theta^2(p) + C_8 \left(\frac{p}{1-p}\right)^M (p-p_c)^\varepsilon\,\theta(p) \cdot \sum_{y \in \mathscr{G}} P_p\{y \text{ is pivotal for } \{O \to \infty\}\}\right]$$

$$\leqslant \frac{2}{p}\left[\theta^2(p) + C_8 \left(\frac{p}{1-p}\right)^M (p-p_c)^\varepsilon\,\theta(p)\,\theta'(p)\right]$$

This will be enough to imply Theorem 1; the factor $[p/(1-p)]^M$ is harmless, since we only have to consider p close to $p_c \in (0, 1)$.

Proof of (3.2). This is quite simple, for let x be the central vertex of F and let $r_1 = (v_0, e_1, \dots, e_n, O)$ be an occupied path on $\mathscr{G}_{\rm pl}$ from a neighbor v_0 of x on $\mathscr{G}_{\rm pl}$ to O, and $r_2 = (u_0, f_1, \dots)$ an occupied path from a neighbor of x to ∞. Also, let $\mathscr{C}^* = (x, g_1^*, w_1^*, \dots, w_{m-1}^*, g_m^*, x)$ be a circuit on $\mathscr{G}^*_{\rm pl}$

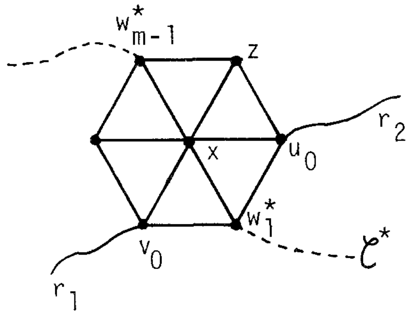


Fig. 8. A face F with a central vertex x pivotal for $\{O \to \infty\}$.

surrounding O and vacant, except at x. Then v_0, u_0, w_1^*, w_{m-1}^* must be vertices of $\mathscr{M}$ (and hence of $\mathscr{G}$ and $\mathscr{G}^*$) on the perimeter of F such that w_1^* and w_{m-1}^* separate v_0 and u_0 (see Fig. 8). If all occupied vertices other than u_0 on the arc of the perimeter of F from w_1^* to w_{m-1}^* through u_0 are made vacant (in Fig. 8 this can only be the vertex z), then u_0 becomes pivotal for $\{O \to \infty\}$. Indeed, if w_1^* and w_{m-1}^* are the only points of $\mathscr{C}^*$ on the perimeter of F, then we can replace $\mathscr{C}^*$ by the new circuit going from u_0 to w_1^* along the perimeter of F, continuing along $\mathscr{C}^*$ to w_{m-1}^*, and then returning to u_0 along the perimeter of F. In general we can do this by replacing w_1^* and w_{m-1}^* by w_k^* and w_l^*, where $w_{k+1}^*,..., w_{l-1}^*$ is a maximal piece of $\mathscr{C}^*$ between w_1^* and w_{m-1}^* that does not intersect the perimeter of F. Thus, if x is pivotal for $\{O \to \infty\}$, then we can make a y on the perimeter of F pivotal by changing at most M vertices from occupied to vacant. (3.2) is immediate from this (cf. Ref. 9, Proposition 8). ∎

Comments to Theorem 2. The form of the proof is very similar to that of Proposition 1. Let $0 \leqslant \varphi_1 < \varphi_2 \leqslant 2\pi$ and let R_0 be the positive x axis, and for $i = 1, 2$ let R_i be the ray

$$R_i = \{(x, y): x = r\cos\varphi_i,\ y = r\sin\varphi_i \text{ for some } r \geqslant 0\}$$

Define the event $\mathscr{F}(i)$ as (see Fig. 9)

$$\begin{aligned} \mathscr{F}(i) &= \{\exists \text{ occupied connection from } R_0 \text{ to } R_2 \text{ in } \mathscr{A}_i \cap \mathscr{S}(\varphi_2)\} \\ \mathscr{E}_j &= \mathscr{F}(3j) \cap \mathscr{F}(3j+2) \end{aligned} \tag{3.3}$$

On $\mathscr{E}_j$ we define $\mathscr{C}_{3j}$ as the "innermost" occupied path with the properties in (3.3) for $i = 3j$, and $\mathscr{D}_{3j+2}$ as the "outermost" occupied path with the properties in (3.3) for $i = 3j+2$. Let

$$J = \{j \geqslant 1: 2^{3j+2} \leqslant L(p) \text{ and } \mathscr{E}_j \text{ occurs}\}$$

$$N = \text{cardinality of } J$$

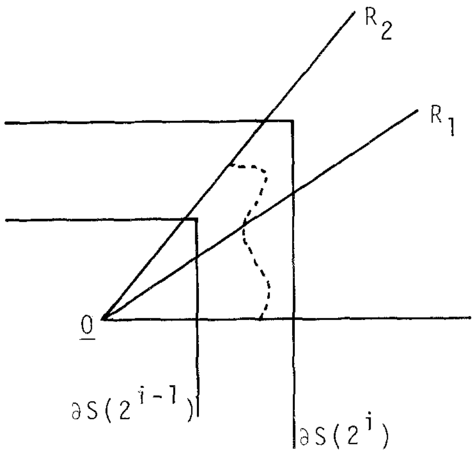


Fig. 9. $\mathscr{F}(i)$ occurs when the dashed path is occupied.

Again, for suitable $C_2 > 0$,

$$\begin{aligned}\theta(p, \varphi_i) &= P_p\{O \to \infty \text{ in } \mathscr{S}(\varphi_i)\} \\ &\leqslant 2P_p\{O \to \infty \text{ in } \mathscr{S}(\varphi_i) \text{ and } N \geqslant -C_2 \log(p - p_c)\} \\ &= 2E_p\{P_p\{O \to \infty \text{ in } \mathscr{S}(\varphi_i) \\ &\quad \text{and } N \geqslant -C_2 \log(p - p_c) \mid J, \mathscr{C}_{3j}, \mathscr{D}_{3j+2}, j \in J\}\} \end{aligned} \tag{3.4}$$

On $\{N < -C_2 \log(p - p_c)\}$ the conditional probability in the last member of (3.4) vanishes. On $\{N \geqslant -C_2 \log(p - p_c)\}$ with $J = \{j(1) < \cdots < j(N)\}$ this conditional probability equals

$$\begin{aligned} &P_p\{O \to \mathscr{C}_{3j(1)} \text{ in } \mathscr{S}(\varphi_i)\} \cdot P_p\{\mathscr{D}_{3j(N)+2} \to \infty \text{ in } \mathscr{S}(\varphi_i)\} \\ &\cdot \prod_{1 \leqslant l \leqslant N} P_p\{\mathscr{C}_{3j(l)} \to \mathscr{D}_{3j(l)+2} \text{ in } \mathscr{S}(\varphi_i)\} \\ &\cdot \prod_{1 \leqslant l \leqslant N-1} P_p\{\mathscr{D}_{3j(l)+2} \to \mathscr{C}_{3j(l+1)} \text{ in } \mathscr{S}(\varphi_i)\} \end{aligned}$$

All probabilities here are calculated with $\mathscr{C}$ and $\mathscr{D}$ viewed as fixed (i.e., nonrandom). One finally obtains (1.6) by proving that there exists a constant $\lambda = \lambda(\mathscr{G}, \varphi_1, \varphi_2) < 1$ such that

$$P_p\{\mathscr{C}_{3j} \to \mathscr{D}_{3j+2} \text{ in } \mathscr{S}(\varphi_1)\} \leqslant \lambda P_p\{\mathscr{C}_{3j} \to \mathscr{D}_{3j+2} \text{ in } \mathscr{S}(\varphi_2)\} \tag{3.5}$$

uniformly in $\mathscr{C}_{3j} \subset \mathscr{A}_{3j}$, $\mathscr{D}_{3j+2} \subset \mathscr{A}_{3j+2}$, and j large enough. The proof of (3.5) is analogous to that of Lemma 4. We merely have to show

$$P_p\{\mathscr{C}_{3j} \to \mathscr{D}_{3j+2} \text{ in } \mathscr{S}(\varphi_2) \text{ but not in } \mathscr{S}(\varphi_1)\} \geqslant C_4 > 0$$

We give no further details. (1.7) follows if one replaces p by p_c and $L(p)$ by n in the above argument. (1.8) and (1.9) follow by taking $\varphi_2 = 2\pi$, $\mathscr{S}(\varphi_2) = \mathbb{R}^2$, and replacing the restriction that a path r stay in $\mathscr{S}(\varphi_1)$ by the restriction that r may not cross the positive x axis.

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